

The Harmonic Oscillator warmup

In this note we will practice a few math tricks that will pop up in our considerations of quantum systems. Given that a Harmonic Oscillator (HO) is probably the most intuitive mechanical system, it's probably a good idea to get to know it better as the reference classical system for our Quantum course. We will study a Quantum Harmonic Oscillator later in the course.

Harmonic oscillator evolution equations in the matrix form

Classical HO (think mass on a spring) is defined by its Hamiltonian function $H(x, p) = p^2/2m + kx^2/2$. Here x, p is the pair of position and momentum variables, respectively, m is the mass and k is the spring stiffness (the larger the k , the harder it is to stretch the spring). The Hamiltonian equations of motion are:

$$\begin{aligned}\dot{x} &= \partial H / \partial p = p/m \\ \dot{p} &= -\partial H / \partial x = -kx\end{aligned}\tag{1}$$

Solving Eq. (1) and using initial conditions $x(t=0) = x_0$ and $p(t=0) = p_0$ allows us to know the values of $x(t), p(t)$ at any given time t . This is a general solution for a classical mechanics problem. A common way to solve Eq. (1) is to eliminate p and obtain a single 2-order equation $\ddot{x} + \omega^2 x = 0$, where $\omega = \sqrt{k/m}$ turns out to be the natural (free) vibration frequency for the oscillator. Here we consider a different route, more appropriate for the rest of the course material. Let us start by rescaling x and p such that they have the same units. This way we would be allowed to add them! After checking the units of t, x, p, m, k, ω one can also check that new variables $X = x\sqrt{m\omega}$ and $P = p/\sqrt{m\omega}$ would have the same units.

[**Exercise 1.1** what are the units of X and P ?]

Now we can consider X and P on equal footing. We can define a vector $\vec{s} = (X, P)^T$ (here T means transpose, that is a column vector) and a matrix $\hat{\Omega} = \omega \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and rewrite our equations of motions as

$$\dot{\vec{s}} = \hat{\Omega} \vec{s}\tag{2}$$

The solution Eq. (2) is given by

$$\vec{s}(t) = \exp(\hat{\Omega}t) \vec{s}(t=0).\tag{3}$$

Here the matrix exponentiation is defined through the power series of the exponential function: $\exp(\hat{\Omega}t) = \hat{I} + \hat{\Omega}t + \hat{\Omega}^2 t^2/2! + \hat{\Omega}^3 t^3/3! + \hat{\Omega}^4 t^4/4! + \dots$ where $\hat{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is called identity matrix.

[**Exercise 1.2** show by explicit multiplication that $\hat{I}^2 = \hat{I}, \hat{\Omega}^2 = -\omega^2 \hat{I}$]

[**Exercise 1.3** show that expression (3) is indeed the solution of Eq. (2) using a direct substitution and the power series expansion]

Oscillator motion is a rotation in the (X, P) -space

Taking into account the results of Exercise 1.2 we obtain a neat expression for the $\exp \hat{\Omega}t$ matrix:

$$\exp(\hat{\Omega}t) = \hat{I} \cos \omega t + \hat{\Omega} \sin \omega t = \begin{pmatrix} \cos \omega t & \sin \omega t \\ -\sin \omega t & \cos \omega t \end{pmatrix}\tag{4}$$

And hence the final solution for the oscillator equations is $\vec{s}(t) = (X(t), P(t))^T$, where $X(t) = X_0 \cos \omega t + P_0 \sin \omega t$, $P(t) = -X_0 \sin \omega t + P_0 \cos \omega t$.

[**Exercise 1.4** proof the above formula for $\exp \hat{\Omega}t$. Hint: recall that $\cos \alpha = 1 - \alpha^2/2! + \alpha^4/4! - \alpha^6/6! + \dots$ and $\sin \alpha = \alpha - \alpha^3/3! + \alpha^5/5! - \dots$

[**Exercise 1.5** Show that the matrix $\exp \hat{\Omega}t$ corresponds to a rotation of a vector in the (X, P) -plane by an angle ωt , clockwise. Hint: set the value of ωt to something convenient, like $\pi/4$ and apply the matrix $\exp \hat{\Omega}t$ to simple vectors such as $(X_0, 0)^T$ or $(0, P_0)^T$].

So (assuming you do the above exercises), we see that the time-evolution of HO after a time t is a rotation of the initial state vector (X_0, P_0) by an angle ωt . One can say this is a rotation at a constant rate, given by $\omega/2\pi$, that is the period of rotation is $2\pi/\omega$. No matter at which point in the (X, P) -plane we start, we will come back to this point in time $T = 2\pi/\omega$ along a circular trajectory. For example, for the simple initial condition $P_0 = 0$, we get the familiar oscillatory in time solution for the evolution of the position $X(t) = X_0 \cos \omega t$. The property that the rate of rotation (the oscillator's period) does not depend on initial conditions is a special one, it's called linearity.

[**Exercise 1.6** Speculate what might be the application of the fact that HO's period does not depend on initial conditions in the watchmaking industry?]

[**Exercise 1.7** Find a matrix $(\exp \hat{\Omega}t)^{-1}$ which is defined as the inverse of $\exp \hat{\Omega}t$. Hint: if the original matrix is a rotation by ωt , what operation would undo that? Verify by performing matrix multiplication $\exp \hat{\Omega}t (\exp \hat{\Omega}t)^{-1}$].

[**Exercise 1.8** show by explicit calculation using Eq. 4 that the quantity $X(t)^2 + P(t)^2 = X_0^2 + P_0^2$, that is, this quantity does not change in time.]

Let us consider one more trick with rotations, this time using complex numbers. We define a complex number $Z = X + iP$. It's real part is X and imaginary part is P , so we can visualize Z as a vector in the (X, P) -plane, just like we did with the state vector $\vec{s}$. How can we write down an operation that rotates this vector by an angle ωt ? Note that any complex number can be written as $Z = |Z| \exp i\alpha$, where α is the angle of the vector with the X-axis. Therefore, to rotate this vector by an angle ωt clockwise, we just need to multiply by $\exp(-i\omega t)$. So let us now similarly introduce the initial state of the oscillator $Z_0 = X_0 + iP_0$ and then the solution for the motion of HO is most compactly written as

$$Z(t) = Z_0 \exp(-i\omega t) \quad (5)$$

Again, what we mean by this equation is that $X(t) + iP(t) = (X_0 + iP_0) \times \exp(-i\omega t)$. Now $X(t) = \text{Re}[(X_0 + iP_0) \times \exp(-i\omega t)]$ and $P(t) = \text{Im}[(X_0 + iP_0) \times \exp(-i\omega t)]$.

[**Exercise 1.9** verify that the last two equations match with the previously obtained solution in Eq. (4)]

Periodically-driven oscillator: resonance

Next we consider the effect of external force on the motion of the HO. We usually say let's "drive" our oscillator. The drive is usually modeled by modifying the Hamiltonian function as $H \rightarrow H - xF(t)$.

[Exercise 1.10] show that the Hamiltonian equations of motion in the presence of drive, in the matrix form, are

$$\dot{\vec{s}} = \hat{\Omega}\vec{s} + \vec{F}(t), \quad (6)$$

where $\vec{F} = (0, f(t))$, where $f(t) = F(t)/\sqrt{m\omega}$

How do we solve the new equations of motion? The trick is the following. Suppose we have a scalar equation, $\dot{s} = \Omega s + F(t)$. In this case, we first take a solution for $F = 0$, which is $s(t) = s(0) \exp(\Omega t)$. Next we take the "initial value" s_0 to be also a function of time, that is $s(t) = s_0(t) \exp(\Omega t)$ and try it as a solution in the case $F \neq 0$.

[Exercise 1.11] show by a direct substitution that $\dot{s}_0 = \exp(-\Omega t)F(t)$, and hence $s_0(t) = \int^t \exp(-\Omega t')F(t')dt' + C$ and $s(t) = \exp(\Omega t)(C + \int^t \exp(-\Omega t')F(t')dt')$. The constant C defines the initial conditions].

Using the reasoning in Exercise 1.9 we can generalize the answer to the matrix form equation for the driven HO Eq. (6):

$$\vec{s}(t) = \exp(\hat{\Omega}t)(\vec{C} + \int^t \exp(-\hat{\Omega}t')\vec{F}(t')dt') \quad (7)$$

The most common driving case is when $f(t) = f_0 \cos \omega_d t$, that is the drive signal contains only one frequency, ω_d . Let us resort again to complex numbers to simplify the calculations by rewriting the expression above by analogy with Eq. (5):

$$Z = X + iP = \exp(-i\omega t)(C + \int_0^t \exp(i\omega t')(0 + if_0 \cos \omega_d t')dt') \quad (8)$$

Now let us consider separately two cases: $\omega_d \neq \omega$ and $\omega_d = \omega$. In the first case, $\omega_d \neq \omega$, we proceed dully with the integration: $Z = \exp(-i\omega t)(C + (if_0/2) \int_0^t \exp(i\omega t')(\exp(i\omega_d t') + \exp(-i\omega_d t'))dt')$. Keep working this out, we get

$$Z = C \exp(-i\omega t) + \exp(i\omega_d t)(f_0/2)/(\omega_d + \omega) + \exp(-i\omega_d t)(f_0/2)/(\omega - \omega_d) \quad (9)$$

The first term corresponds to free oscillations (rotations in the X-P plane) at the oscillator's natural frequency ω . This is a result of it's initial displacement at some point in time, set by constant C . The other terms are more interesting as they are caused entirely by the external periodic force. Let us ignore the effect of the free oscillations by setting $C = 0$:

$$Z = X + iP = \exp(i\omega_d t)(f_0/2)/(\omega_d + \omega) + \exp(-i\omega_d t)(f_0/2)/(\omega - \omega_d) \quad (10)$$

We can see that now the solution for the state vector of the driven oscillator consists of two components. The first one is a vector of length $f_0/(2(\omega_d + \omega))$ rotating counterclockwise at a rate ω_d and a second vector of larger length $f_0/(2|\omega - \omega_d|)$, rotating clockwise also at a rate ω_d . Near the resonance condition, $|\omega - \omega_d| \ll \omega$ we can neglect the counterclockwise rotating vector as it's length is much smaller than that of the clockwise rotating vector. Therefore to a good approximation, the solution near the resonance condition is

$$X + iP = \frac{f_0}{2(\omega - \omega_d)} \exp(-i\omega_d t) \quad (11)$$

The difference with the free oscillations case (Eq. 5) is that now HO oscillates at the drive frequency but with an amplitude which gets the larger the closer the drive frequency is to ω . This is the phenomenon of resonance: when an oscillator is driven at a frequency close to its free oscillations frequency, it becomes very efficient at taking the energy from the drive and oscillate at a larger amplitude.

So what happens on resonance, when $\omega_d = \omega$? In this case we get a division by zero in Eq. (11), which means we should have been a bit more careful in working out the integral. Let's go back to it, set $\omega_d = \omega$ and it looks that now we get $Z = \exp(-i\omega t)(C + (if_0/2) \int^t (\exp(2i\omega t') + 1) dt')$.

[Exercise 1.12 Work out the expression for $Z(t) = X(t) + iP(t)$ at $\omega_d = \omega$ and plot $X(t)$ as well as a t-parametric plot in the X-P plane, starting from $t = 0$ to $t = 17 \times 2\pi/\omega$. Assume that at $t = 0$, $X = 0$, and $P = 0$. Describe in words what happens with an oscillator driven exactly on resonance. After you are done, look up online and compare with your result].

Answer 1.12

$$Z(t) = X + iP = i(f_0/2)t \exp(-i\omega t) + i(f_0/2) \sin(\omega t), \quad (12)$$

from which we get $X(t) = (f_0/2)t \sin(\omega t)$ and $P(t) = (f_0/2)t \cos(\omega t) + (f_0/2\omega) \sin(\omega t)$. Now these equations no longer describe a rotation at a rate ω , because the value of $X^2 + P^2$ now increases in time as t^2 . A resonant drive causes the oscillator to accept the energy from the drive without limits. By contrast, when we are slightly off-resonant, $\omega_d - \omega \neq 0$, there is a limit to this process, and the oscillations amplitude for X and P stabilizes, as written in Eq. (11).